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ADDENDUM

Normal ordering expansion of *n*-dimensional radial coordinate operators gained by virtue of the IWOP technique

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Abstract
By virtue of the technique of integration within an ordered product of operators we derive the normal ordering expansion of the power of radial coordinate operators in the n-dimensional coordinate space. The use of Bessel function has greatly simplified the calculation. Moreover, the use of Kummer's first formula for the confluent hypergeometric function makes the result neat and concise.

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1. Introduction

In a recent paper [1] we have derived the normal ordering expansion of the Dirac's radial momentum operator [2], and the operator identities of the power of the radial coordinate operator $\hat{r}$, $\hat{r}$ corresponds to the radius value $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. The derivation is proceeded by virtue of the technique of integration within an ordered product (IWOP) of operators [3]. For example, we have shown that the normal ordering of $\hat{r}^k$ ($k = 2m$ is even)

$$\hat{r}^{2m} = \sum_{l=0}^{m} \frac{(2m+1)}{4^l(2m+1-2l)!l!} : |\hat{r}|^{2m-2l} : \tag{1}$$

where :: denotes normal ordering, $\hat{r}^2 = X_1^2 + X_2^2 + X_3^2$, X_i is related to bosonic creation operator $a^\dagger$ and annihilation operator a by

$$X_i = \frac{1}{\sqrt{2}}\left(a_i^\dagger + a_i\right). \tag{2}$$

An interesting question thus naturally arises: formula (1) holds in the three-dimensional coordinate case, and r is $SO(3)$ rotation invariant. How about if r is $SO(n)$ rotation invariant, which means when $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, $\hat{r}^2 = X_1^2 + X_2^2 + \cdots + X_n^2$, then how to modify formula (1)? To put it in another way, what is the normal ordering expansion of operator $\hat{r}^k$ defined in n-dimensional coordinate space? In the following we shall employ the IWOP technique to discuss it.

We study the Hermitian radius operator $\hat{r}^k$ in n-dimensional coordinate space via the equation

$$\hat{r}^k = \int \mathrm{d}^n\vec{x}\, |\vec{x}\rangle\langle\vec{x}| r^k \tag{3}$$

where $|\vec{x}\rangle$ is the n-dimensional coordinate eigenvector $|\vec{x}\rangle = |x_1\rangle|x_2\rangle \cdots |x_n\rangle$. As the first step, we use the IWOP technique to truely perform the integration of $\int \mathrm{d}^n\vec{x}\, |\vec{x}\rangle\langle\vec{x}|$ in spherical polar coordinates to confirm both the n-dimensional completeness relation and the feasibility of the IWOP technique.

1.1. Preparation for the integration in n-dimensional spherical polar coordinate

The Fock space expansion of the n-dimensional coordinate eigenvector is

$$|\vec{x}\rangle = \pi^{-\frac{n}{4}} \exp\left\{\sum_{i=1}^{n}\left[-\frac{1}{2}x_i^2 + \sqrt{2}x_i a_i^\dagger - \frac{1}{2}a_i^{\dagger^2}\right]\right\}|\vec{0}\rangle. \tag{4}$$

Using the normal ordering of the n-mode vacuum state projector

$$|\vec{0}\rangle\langle\vec{0}| =: \exp\left[-\sum_{i=1}^{n} a_i^\dagger a_i\right]: \tag{5}$$

and defining $r^2 = \sum_{i=1}^{n} x_i^2$, $\vec{x} = (x_1, x_2, \ldots, x_n)$, $X_i = \frac{1}{\sqrt{2}}\left(a_i + a_i^\dagger\right)$, $\vec{X} = (X_1, X_2, \ldots, X_n)$, we have

$$\begin{aligned}|\vec{x}\rangle\langle\vec{x}| &= \pi^{-\frac{n}{2}} : \exp\left\{\sum_{i=1}^{n}\left[-x_i^2 + \sqrt{2}x_i\left(a_i + a_i^\dagger\right) - \frac{1}{2}\left(a_i^{\dagger^2} + a_i^2\right) - a_i^\dagger a_i\right]\right\}: \\ &= \pi^{-\frac{n}{2}} : \exp\{-r^2 + 2\vec{x}\cdot\vec{X} - \vec{X}^2\} : .\end{aligned} \tag{6}$$

$\vec{X}$ is now within the normal ordering symbol :: and can be treated as a c-number vector when one performs an integral over $\vec{x}$ in equation (6). For simplifying the integration, we rotate the frame of coordinate system $\vec{x}$ to $\vec{x'}$, a new frame of coordinates whose one component x_1' axis directs along the $\vec{X}$ vector, so that $\vec{x}\cdot\vec{X} = x_1'\sqrt{X_1^2 + X_2^2 + \cdots + X_n^2}$. This rotation, keeping both the radius $r = r' = \sqrt{\sum_{i=1}^{n} x_i'^2}$ and the integration measure invariant, is an orthogonal transformation with Jacobian being 1. Then equation (6) becomes

$$\int \mathrm{d}^n\vec{x}\, |\vec{x}\rangle\langle\vec{x}| = \pi^{-\frac{n}{2}} : \int \mathrm{d}^n\vec{x'} \exp\{-r'^2 + 2|\hat{r}|x_1' - |\hat{r}|^2\} : \tag{7}$$

where we have defined

$$|\hat{r}| = \sqrt{X_1^2 + X_2^2 + \cdots + X_n^2}.$$

Now we are ready to integrate $\mathrm{d}^n\vec{x'}$ of (7) in its n-dimensional spherical polar coordinate.

1.2. The one-dimensional radial coordinate integration involved in $\int \mathrm{d}^n\vec{x}\, |\vec{x}\rangle\langle\vec{x}|$

Firstly, we transform $\mathrm{d}^n\vec{x'}$ to the volume element in spherical polar coordinates by setting

$$\begin{aligned}
x_1' &= r'\cos\theta_1\\
x_2' &= r'\sin\theta_1\cos\theta_2\\
&\cdots\\
x_{n-1}' &= r'\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\theta_{n-1}\\
x_n' &= r'\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\theta_{n-1}
\end{aligned}$$

where $0 \leqslant \theta_1 < \pi, \ldots, 0 \leqslant \theta_{n-2} < \pi, 0 \leqslant \theta_{n-1} < 2\pi$. The Jacobian for the transformation is

$$J = \frac{\partial(x_1', x_2', \ldots, x_n')}{\partial(r', \theta_1, \ldots, \theta_{n-1})} = r'^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2}.$$

Therefore,

$$\begin{aligned}
\int \mathrm{d}^n\vec{x}\, |\vec{x}\rangle\langle\vec{x}| &= \pi^{-\frac{n}{2}}\int_0^\infty\int_0^\pi\cdots\int_0^\pi\int_0^{2\pi}\mathrm{d}r'\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{n-2}\,\mathrm{d}\theta_{n-1}\\
&\quad\times r'^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2\cdots\sin\theta_{n-2} : \exp\{-r'^2 + 2|\hat{r}|r'\cos\theta_1 - |\hat{r}|^2\} : . \qquad (8)
\end{aligned}$$

Using the well-known formula

$$\int_0^\pi \sin^m\theta\,\mathrm{d}\theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}+1\right)}$$

where Γ is the Gamma function, and adopting the IWOP technique we perform the integrations over the azimuth angles $\theta_2, \theta_3, \ldots, \theta_{n-1}$ in (8) one by one, and obtain

$$\int \mathrm{d}^n\vec{x}\, |\vec{x}\rangle\langle\vec{x}| = \frac{2}{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} : \int_0^\infty \mathrm{d}r'\, r'^{n-1}\,\mathrm{e}^{-r'^2}\int_0^\pi \mathrm{d}\theta_1\sin^{n-2}\theta_1\,\mathrm{e}^{2|\hat{r}|r'\cos\theta_1}\,\mathrm{e}^{-|\hat{r}|^2} : . \qquad (9)$$

Then we recall the Poisson integration formulation [7]

$$J_\upsilon(z) = \frac{(z/2)^\upsilon}{\sqrt{\pi}\,\Gamma(\upsilon+1/2)}\int_0^\pi \exp(\mathrm{i}z\cos\theta)\sin^{2\upsilon}\theta\,\mathrm{d}\theta \qquad \mathrm{Re}(\nu) > -\frac{1}{2} \qquad (10)$$

where $J_\upsilon(z)$ is the Bessel function, and z and ν are complex numbers. To further carry out the integration over $\mathrm{d}\theta_1$ in (9) we let $-2\mathrm{i}|\hat{r}|r' = z$, $\frac{n}{2} - 1 = \nu$ (note $n \geqslant 3$), and use (10) to obtain

$$\int_0^\pi \mathrm{d}\theta_1\sin^{n-2}\theta_1\,\mathrm{e}^{2|\hat{r}|r'\cos\theta_1} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{(-\mathrm{i}|\hat{r}|r')^{\frac{n}{2}-1}}J_{\frac{n}{2}-1}(-2\mathrm{i}|\hat{r}|r'). \qquad (11)$$

Substituting (11) into (9) yields

$$\int \mathrm{d}^n\vec{x}\, |\vec{x}\rangle\langle\vec{x}| = 2 : \int_0^\infty r'^{\frac{n}{2}}\,\mathrm{e}^{-r'^2}J_{\frac{n}{2}-1}(-2\mathrm{i}|\hat{r}|r')\,\mathrm{d}r'\frac{\mathrm{e}^{-|\hat{r}|^2}}{(-\mathrm{i}|\hat{r}|)^{\frac{n}{2}-1}} : . \qquad (12)$$

Using the definition of Bessel function [7]

$$J_\nu(z) = \sum_{l=0}^\infty \frac{(-1)^l z^{2l+\nu}}{l!\,\Gamma(\nu+l+1)} \qquad (13)$$

and that of the Gamma function

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t}t^{z-1}\,\mathrm{d}t \qquad \mathrm{Re}(z) > 0 \qquad (14)$$

we obtain

$$\int \mathrm{d}^n \vec{x}\, |\vec{x}\rangle\langle\vec{x}| = 2 : \int_0^\infty r'^{\frac{n}{2}}\, \mathrm{e}^{-r'^2} \left(\sum_{l=0}^{\infty} \frac{(-1)^l}{l!\Gamma\left(\frac{n}{2}+l\right)} (-\mathrm{i}|\hat{r}|r')^{2l+\frac{n}{2}-1} \right) \mathrm{d}r' \frac{\mathrm{e}^{-|\hat{r}|^2}}{(-\mathrm{i}|\hat{r}|)^{\frac{n}{2}-1}} :$$

$$=: \sum_{l=0}^{\infty} \frac{1}{l!\Gamma\left(\frac{n}{2}+l\right)} \left(\int_0^\infty \mathrm{e}^{-\lambda} \lambda^{l-1+n/2}\, \mathrm{d}\lambda \right) |\hat{r}|^{2l} \mathrm{e}^{-|\hat{r}|^2} :$$

$$=: \sum_{l=0}^{\infty} \frac{1}{l!} (|\hat{r}|^2)^l\, \mathrm{e}^{-|\hat{r}|^2} := 1. \tag{15}$$

By far, we have used the IWOP technique to confirm the n-dimensional completeness relation $\int \mathrm{d}^n\vec{x}\, |\vec{x}\rangle\langle\vec{x}|$ in spherical polar coordinates.

2. The normal product form of the radial coordinate operator $\hat{r}^k$ in n-dimensional coordinate space

Now using (3) and (15) as well as the IWOP technique, we can derive the normal product form of the radial coordinate operator $\hat{r}^k$,

$$\hat{r}^k = 2 : \left(\int_0^\infty \mathrm{d}r'\, r'^{\frac{n}{2}+k}\, \mathrm{e}^{-r'^2} J_{\frac{n}{2}-1}(-2\mathrm{i}|\hat{r}|r') \right) \frac{\mathrm{e}^{-|\hat{r}|^2}}{(-\mathrm{i}|\hat{r}|)^{\frac{n}{2}-1}} :$$

$$=: \sum_{l=0}^{\infty} \frac{1}{l!\Gamma\left(\frac{n}{2}+l\right)} \left(\int_0^\infty \mathrm{e}^{-\lambda} \lambda^{l-1+\frac{n+k}{2}}\, \mathrm{d}\lambda \right) |\hat{r}|^{2l}\, \mathrm{e}^{-|\hat{r}|^2} :$$

$$=: \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma\left(l+\frac{n+k}{2}\right)}{l!\Gamma\left(\frac{n}{2}+l\right)} |\hat{r}|^{2l}\, \mathrm{e}^{-|\hat{r}|^2} : . \tag{16}$$

With the help of the recurrence relation of the Gamma function [7]

$$\Gamma(z+n) = (z+n-1)(z+n-2)\cdots z\Gamma(z) \qquad (n \text{ is an integer}, n \geqslant 1)$$

equation (16) becomes

$$\hat{r}^k =: \frac{\Gamma\left(\frac{n+k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \sum_{l=0}^{\infty} \frac{\left(\frac{n+k}{2}\right)_l}{l!\left(\frac{n}{2}\right)_l} |\hat{r}|^{2l}\, \mathrm{e}^{-|\hat{r}|^2} : \tag{17}$$

where we have defined

$$(\lambda)_0 = 1, \ldots, (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad n \geqslant 1.$$

For simplicity, it is worthwhile to rewrite the above result in terms of the confluent hypergeometric function $F(\alpha;\gamma;z)$,

$$F(\alpha;\gamma;z) = \sum_{l=0}^{\infty} \frac{(\alpha)_l}{l!\,(\gamma)_l} z^l. \tag{18}$$

Thus

$$\hat{r}^k =: \frac{\Gamma\left(\frac{n+k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} F\left(\frac{n+k}{2}; \frac{n}{2}; |\hat{r}|^2 \right) \mathrm{e}^{-|\hat{r}|^2} : . \tag{19}$$

Further, using Kummer's first formula [8]

$$F(\alpha;\gamma;z)=\mathrm{e}^{z}F(\gamma-\alpha;\gamma;-z) \tag{20}$$

we can further simplify (19) as

$$\hat{r}^k =: \frac{\Gamma\left(\frac{n+k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}F\left(-\frac{k}{2};\frac{n}{2};-|\hat{r}|^2\right):. \tag{21}$$

By far, we have derived the normally ordered expansion of power of radial coordinate operator $\hat{r}^k$ in n-dimensional coordinate space. Note that n denotes the dimension of coordinate space.

In particular, when $n=3$, (19) reduces to (note $(-m)_{m+1}=(-m)_{m+2}=\cdots=0$)

$$\begin{aligned}\hat{r}^{2m} &=: \frac{\Gamma(m+\frac{3}{2})}{\Gamma\left(\frac{3}{2}\right)}F\left(-m;\frac{3}{2};-|\hat{r}|^2\right): \;=\; : \frac{(2m+1)!!}{2^m}\sum_{l=0}^{m}\frac{(-m)_l}{l!\left(\frac{3}{2}\right)_l}(-|\hat{r}|^2)^l :\\ &=: \sum_{l'=0}^{m}\frac{(2m+1)!!2^{-l'}m!}{l'!(m-l')!(2m-2l'+1)!!}|\hat{r}|^{2m-2l'} :\\ &=: \sum_{l'=0}^{m}\frac{(2m+1)!}{4^{l'}l'!(2m-2l'+1)!}|\hat{r}|^{2m-2l'} :\end{aligned} \tag{22}$$

as expected. When $n=4$, (21) reduces to

$$\begin{aligned}\hat{r}^{2m} &=: (m+1)!\sum_{l=0}^{m}\frac{(-m)_l}{l!(2)_l}(-|\hat{r}|^2)^l :\\ &=: (m+1)!\sum_{l'=0}^{m}\frac{(-m)_{m-l'}}{(m-l')!(2)_{m-l'}}(-|\hat{r}|^2)^{m-l'} :\\ &=: \sum_{l'=0}^{m}\frac{(m+1)!m!}{l'!(m-l')!(m-l'+1)!}|\hat{r}|^{2m-2l'} :.\end{aligned} \tag{23}$$

Equations (23) and (22) are quite different.

In summary, we have derived the normal product form of the radial coordinate operator $\hat{r}^k$ in the n-dimensional coordinate space, which is a non-trivial generalization of [1]. Comparing the derivation of this work with the integration procedures in our previous paper [1], we conclude that the use of Bessel function defined in (18) has greatly simplified the calculation. Moreover, the use of Kummer's first formula for the confluent hypergeometric function makes the result neat and concise.

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